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Below-Threshold Effects for the Two Particle Discrete Schrödinger Operator on a Lattice

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Abstract—We consider the family of Schrödinger operators $H_{\gamma\lambda}(K)$, which are associated with the Hamiltonian of a system of two identical bosons on the d-dimensional lattice $\mathbb{Z}^d$, where $d \geq 3$, with interactions on each site and between nearest-neighbor sites with strengths $\gamma \in \mathbb{R}_-$ and $\lambda \in \mathbb{R}_-$, respectively. Here, $K \in \mathbb{T}^d$ is a fixed quasi-momentum of the particles. We first partition the (γ, λ)-plane into connected components $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{C}_j$, $j = 0,1,2$. Further, we establish below-threshold effects for $H_{\gamma\lambda}(0)$ on the boundaries of the connected components $\partial\mathcal{S}_0$ and $\partial\mathcal{C}_j$, $j = 0, 2$.

Keywords: integer lattice, Hamiltonian of a two-particle system, discrete Schrödinger operator, essential spectrum, asymptotic, Fredholm determinant, threshold resonance, threshold eigenvalue

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INTRODUCTION

Lattice models play an important role in various fields of physics [1–3]. In recent years, the study of ultracold systems of several atoms in optical lattices has become very popular, because these systems have easily controllable parameters such as lattice geometry and dimensionality, particle masses, two-body potentials, temperature, etc. (see, e.g., [4–7] and references therein). Unlike traditional condensed matter systems, where stable composite objects are typically formed by attractive forces, the controllability of ultracold atomic systems in an optical lattice makes it possible to experimentally observe a stable pair of ultracold atoms bound by repulsion [8, 9]. In all these observations, Bose–Hubbard Hamiltonians became the link between the theoretical basis and the experimental results.

The main difficulty in solving the Bose–Hubbard Hamiltonian, even with site interactions, is related to tunneling, that is, the kinetic energy required for a boson to move from site to site, because it is highly nonlocal. Moreover, unlike its continuous counterpart, the lattice Hamiltonian corresponding to short-interacting systems of particle pairs is not translation-invariant; hence, separation of the lattice Hamiltonian associated with the motion of the center of mass is impossible. However, the translation invariance of the Hamiltonian in d-dimensional lattice $\mathbb{Z}^d$ allows using the Floquet–Bloch decomposition: the underlying Hilbert space $\ell^{2,s}(\mathbb{Z}^d \times \mathbb{Z}^d)$ and the complete two-particle Hamiltonian $\widehat{\mathbb{H}}_{\gamma\lambda}$ are represented as a direct von Neumann integral

$$\ell^{2,s}(\mathbb{Z}^d \times \mathbb{Z}^d) \simeq \int\limits_{K\in\mathbb{T}^d} \oplus\, \ell^{2,e}(\mathbb{Z}^d)\mathrm{d}K, \qquad \widehat{\mathbb{H}}_{\gamma\lambda} \simeq \int\limits_{K\in\mathbb{T}^d} \oplus\, \widehat{H}_{\gamma\lambda}(K)\mathrm{d}K,$$

associated with the representation of a discrete group $\mathbb{Z}^d$ of shift operators, where $\mathbb{T}^d$ is a d-dimensional torus. The layered operator $\widehat{H}_{\gamma\lambda}(K)$ nontrivially depends on the so-called *quasi-momentum* $K \in \mathbb{T}^d$ (see, e.g., [1, 10–15]).

Note that some spectral properties of two-particle and three-particle Schrödinger operators were studied in [16–20].

The conditions for the existence of eigenvalues in the problem of point interaction of two arbitrary particles for a two-dimensional space are studied in [21, 22], where the Laplace operators are considered to be operators with some domains of definition that vanish on straight lines in $\mathbb{R}$. For a single-particle discrete Schrödinger operator on a three-dimensional lattice, bound states emerge from the essential spectrum either as a threshold bound state or as a threshold resonance [23]. However, the results of works [24, 25] show that in the case $d = 2$, although antisymmetric bound states arise from threshold eigenvalues, all symmetric bound states of the two-particle discrete Schrödinger operator $\widehat{H}_{\gamma\lambda}(0)$ arise from the peculiarity of the associated Fredholm determinant at thresholds, namely, the associated Fredholm determinant cannot be analytically (even continuously) continued to the edges of the essential spectrum. This result is also true in $d = 1$ [26]. In [27], sufficient conditions were found for the existence of eigenvalues of a Schrödinger-type operator that lie either to the left of the lower part of the essential spectrum or to the right of its upper part. The spectral properties of the two-particle Schrödinger operator are studied by constructing invariant subspaces for $d = 3$ in [28] and for $d \geqslant 1$ in [29].

In this paper, we consider a family of discrete Schrödinger operators

$$\widehat{H}_{\gamma\lambda}(K) := \widehat{H}_0(K) + \widehat{V}_{\gamma\lambda}, \quad K \in \mathbb{T}^d,$$

associated with the Hamiltonian $\widehat{\mathbb{H}}_{\gamma\lambda}$ of the system of two identical bosons on d-dimensional lattice $\mathbb{Z}^d, d \geq 3$, interacting at the node and the nearest neighboring nodes with the values $\gamma \in \mathbb{R}_-$ and $\lambda \in \mathbb{R}_-$, respectively. To our knowledge, this is a new, exactly solvable model for which the exact number of eigenvalues and their locations can be found, as well as the exact lower and upper bounds on the number of eigenvalues for all values of pairwise interactions $\gamma, \lambda \in \mathbb{R}_-$. We introduce Fredholm determinants $\Delta_{\gamma\lambda}^{(s)}(z)$ and $\Delta_{\lambda}^{(a)}(z)$, associated with contraction $\widehat{H}_{\gamma\lambda}(0)$ into subspaces of symmetric and antisymmetric functions of the space $L^{2,e}(\mathbb{T}^d)$, respectively. This allows setting the number of eigenvalues below the essential spectrum $\widehat{H}_{\gamma\lambda}(0)$. First, we define the asymptotics $\Delta_{\gamma\lambda}^{(s)}(z)$ and $\Delta_{\lambda}^{(a)}(z)$ as $z \nearrow \mathcal{E}_{\min}(0)$, where $\mathcal{E}_{\min}(0) = 0$ is the lower limit of the essential spectrum. Next, in accordance with the zeros of the asymptotic coefficients, we divide the plane (γ, λ) into connected components $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{C}_j$, $j = 0, 1, 2$. Note that the exact number of eigenvalues $H_{\gamma\lambda}(0)$ and their location were investigated in [30] for the case when (γ, λ) are interior points of these partitions. The main result of this work is the proof of the existence of a threshold eigenvalue or threshold resonance $\widehat{H}_{\gamma\lambda}(0)$ at the boundaries of connected components $\partial\mathcal{S}_0$ and $\partial\mathcal{C}_j$, $j = 0, 2$.

1. TWO-PARTICLE HAMILTONIAN ON A LATTICE

1.1. Coordinate Representation

Let $\mathbb{Z}^d$ be a d-dimensional lattice and $\ell^{2,s}(\mathbb{Z}^d \times \mathbb{Z}^d)$ be the Hilbert space of square-summable symmetric functions on $\mathbb{Z}^d \times \mathbb{Z}^d$.

In the coordinate representation, the two-particle Hamiltonian $\widehat{\mathbb{H}}_{\gamma\lambda}$ corresponding to a system of two bosons interacting at a node and the nearest neighboring nodes with potential $\hat{v}_{\gamma\lambda}$ is a bounded self-adjoint operator acting in the space $\ell^{2,s}(\mathbb{Z}^d \times \mathbb{Z}^d)$ by the formula

$$\widehat{\mathbb{H}}_{\gamma\lambda} = \widehat{\mathbb{H}}_0 + \widehat{\mathbb{V}}_{\gamma\lambda}, \quad \gamma, \lambda \in \mathbb{R}_-.$$

Here is the free Hamiltonian $\widehat{\mathbb{H}}_0$ of the system of two identical particles (bosons), acts in $\ell^{2,s}(\mathbb{Z}^d \times \mathbb{Z}^d)$, and is given by the formula

$$\widehat{\mathbb{H}}_0 \hat{f}(x, y) = \sum_{n \in \mathbb{Z}^d} \hat{\epsilon}(x - n)\hat{f}(n, y) + \sum_{n \in \mathbb{Z}^d} \hat{\epsilon}(y - n)\hat{f}(x, n),$$

where

$$\hat{\epsilon}(s) = \begin{cases} d, & \text{if } |s| = 0; \\ -\dfrac{1}{2}, & \text{if } |s| = 1; \\ 0, & \text{if } |s| > 1, \end{cases}$$

$|s| = \sum_{i=1}^{d} |s_i|$, $s = (s_1, \ldots, s_d) \in \mathbb{Z}^d$.

The interaction potential $\widehat{\mathbb{V}}_{\gamma\lambda}$ is a multiplication operator defined by the formula

$$\widehat{\mathbb{V}}_{\gamma\lambda} \hat{f}(x, y) = \hat{v}_{\gamma\lambda}(x - y)\hat{f}(x, y),$$

where

$$\hat{v}_{\gamma\lambda}(s) = \begin{cases} \gamma, & \text{if } |s| = 0; \\ \dfrac{\lambda}{2}, & \text{if } |s| = 1; \\ 0, & \text{if } |s| > 1. \end{cases}$$

1.2. Impulse Representation

Let $\mathbb{T}^d$ be a d-dimensional torus, which is defined as $(\mathbb{R}/2\pi\mathbb{Z})^d \equiv [-\pi, \pi)^d$. This is the double Pontryagin group of $\mathbb{Z}^d$, and it is equipped with the Haar measure dp.

Let $L^{2,s}(\mathbb{T}^d \times \mathbb{T}^d)$ be the Hilbert space of square-integrable symmetric functions on $\mathbb{T}^d \times \mathbb{T}^d$ and suppose that $L^{2,e}(\mathbb{T}^d)$ is the subspace of even functions on $\mathbb{T}^d$.

Using the standard Fourier transform and Floquet–Bloch decomposition methods [10, 30], we obtain a self-adjoint layer operator $H_{\gamma\lambda}(K)$, $K \in \mathbb{T}^d$, acting in $L^{2,e}(\mathbb{T}^d)$ by the formula

$$H_{\gamma\lambda}(K) := H_0(K) + V_{\gamma\lambda},$$

where $H_0(K)$ is an unperturbed operator defined as the operator of multiplication by a function

$$\mathcal{E}_K(p) := 2\sum_{i=1}^{d}\left(1 - \cos\frac{K_i}{2}\cos p_i\right).$$

The disturbance $V_{\gamma\lambda}$ is determined by the expression

$$V_{\gamma\lambda} f(p) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\left(\gamma + \lambda\sum_{i=1}^{d}\cos p_i \cos q_i\right) f(q) \mathrm{d}q.$$

In the literature, the parameter $K \in \mathbb{T}^d$ is called the *two-particle quasi-momentum*, and the layer $H_{\gamma\lambda}(K)$ is referred to as the *discrete Schrödinger operator* associated with the two-particle Hamiltonian $\widehat{\mathbb{H}}_{\gamma\lambda}$.

1.3. Essential Spectrum of the Discrete Schrödinger Operator

Because the rank $V_{\gamma\lambda}$ does not exceed $d + 1$, according to Weyl's theorem, the essential spectrum $H_{\gamma\lambda}(K)$ coincides with the spectrum $H_0(K)$ for any $K \in \mathbb{T}^d$, that is,

$$\sigma_{\text{ess}}(H_{\gamma\lambda}(K)) = \sigma(H_0(K)) = [\mathcal{E}_{\min}(K), \mathcal{E}_{\max}(K)],$$

where

$$\mathcal{E}_{\min}(K) := \min_{p \in \mathbb{T}^d} \mathcal{E}_K(p) = 2\sum_{i=1}^{d}\left(1 - \cos\frac{K_i}{2}\right) \geqslant 0 = \mathcal{E}_{\min}(0),$$

$$\mathcal{E}_{\max}(K) := \max_{p\in\mathbb{T}^d} \mathcal{E}_K(p) = 2\sum_{i=1}^{d}\left(1+\cos\frac{K_i}{2}\right) \leqslant 4d = \mathcal{E}_{\max}(0).$$

Let $K = 0$. The function $\mathcal{E}_0(p) = 2\sum_{i=1}^{d}(1-\cos p_i)$ is symmetric with respect to permutations of variables p_i and p_j, $i, j = 1,2,\ldots,d$. Therefore, all integrals

$$\int_{\mathbb{T}^d}\frac{\cos p_i\,\mathrm{d}p}{\mathcal{E}_0(p)-z}, \quad \int_{\mathbb{T}^d}\frac{\cos^2 p_i\,\mathrm{d}p}{\mathcal{E}_0(p)-z}, \quad \int_{\mathbb{T}^d}\frac{\cos p_i\cos p_j\,\mathrm{d}p}{\mathcal{E}_0(p)-z}$$

are independent of $i, j = 1,2,\ldots,d$, $i \neq j$. Let us introduce the notation

$$a(z) := \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\frac{\mathrm{d}p}{\mathcal{E}_0(p)-z}, \qquad b(z) := \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\frac{\cos p_i\,\mathrm{d}p}{\mathcal{E}_0(p)-z},$$

$$c(z) := \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\frac{\cos^2 p_i\,\mathrm{d}p}{\mathcal{E}_0(p)-z}, \qquad e(z) := \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\frac{\cos p_i\cos p_j\,\mathrm{d}p}{\mathcal{E}_0(p)-z},$$

$i, j = 1,2,\ldots,d, i \neq j$.

Lemma 1. *The functions* $a(\cdot)$, $b(\cdot)$, $c(\cdot)$, *and* $e(\cdot)$ *are analytic in* $\mathbb{C}\backslash[0,4d]$. *In addition, they are increasing and positive on* $(-\infty,0)$.

Lemma 1 can be proved similarly to Proposition 1 in [23]. For any $d \geqslant 3$ the following finite limits hold:

$$\lim_{z\nearrow 0} a(z) = a(0) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\frac{\mathrm{d}p}{\mathcal{E}_0(p)}, \qquad \lim_{z\nearrow 0} b(z) = b(0) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\frac{\cos p_i\mathrm{d}p}{\mathcal{E}_0(p)},$$

$$\lim_{z\nearrow 0} c(z) = c(0) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\frac{\cos^2 p_i\mathrm{d}p}{\mathcal{E}_0(p)}, \quad \lim_{z\nearrow 0} e(z) = e(0) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\frac{\cos p_i\cos p_j\mathrm{d}p}{\mathcal{E}_0(p)}.$$

2. MAIN RESULTS

Let

$$\lambda_0 = (e(0)-c(0))^{-1} < 0.$$

We define the following connected components in the plane (γ,λ) (Fig. 1):

$$\mathcal{S}_1 := \left\{(\gamma,\lambda)\in\mathbb{R}^2 : \lambda < \lambda_0\right\},$$
$$\mathcal{S}_0 := \left\{(\gamma,\lambda)\in\mathbb{R}^2 : \lambda_0 < \lambda\right\},$$

$$\mathcal{C}_0 := \left\{(\gamma,\lambda)\in\mathbb{R}^2 : 1+\gamma a(0)+\lambda\left(d+\frac{\gamma}{2}\right)b(0) > 0,\ \lambda > -2\frac{a(0)}{b(0)}\right\},$$
$$\mathcal{C}_1 := \left\{(\gamma,\lambda)\in\mathbb{R}^2 : 1+\gamma a(0)+\lambda\left(d+\frac{\gamma}{2}\right)b(0) < 0\right\},$$
$$\mathcal{C}_2 := \left\{(\gamma,\lambda)\in\mathbb{R}^2 : 1+\gamma a(0)+\lambda\left(d+\frac{\gamma}{2}\right)b(0) > 0,\ \lambda < -2\frac{a(0)}{b(0)}\right\}.$$

Note that in each of the components $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{C}_j$, $j = 0,1,2$, the number of eigenvalues of the operators $H^a_\lambda(0)$ and $H^s_{\gamma\lambda}(0)$ lying below their essential spectra, respectively, remains constant ([31], Theorem 3.2).

Moreover, in each of the above mentioned components $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{C}_j$, $j = 0,1,2$, the exact number of eigenvalues of the operators $H^a_\lambda(0)$ and $H^s_{\gamma\lambda}(0)$ is specified, which lie below their essential spectra. This is established in the following theorem and also proved in ([30], Theorem 1).

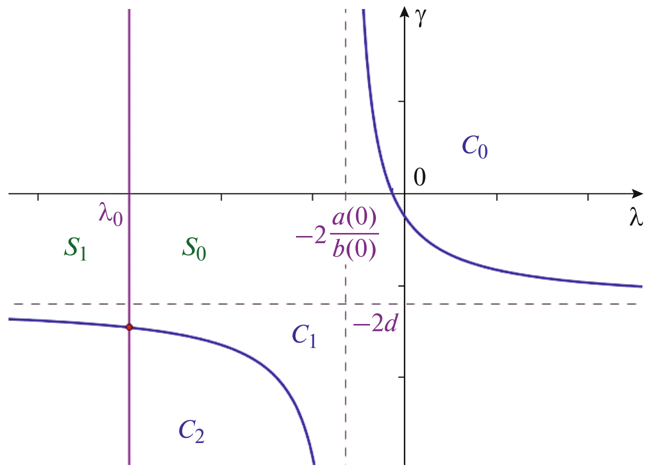

Fig. 1. Connected components $\mathcal{S}_0$, $\mathcal{S}_1$, $\mathcal{C}_j$, $j = 0,1,2$.

Theorem 1. *Let $K = 0$ and $d \geqslant 3$. The following statements are true*:

• *If $(\gamma,\lambda) \in \overline{\mathcal{S}_0}$, then the operator $H^a_\lambda(0)$ has no eigenvalues below the essential spectrum, where $\overline{A}$ is the closure of the set A.*

• *If $(\gamma,\lambda) \in \mathcal{S}_1$, then the operator $H^a_\lambda(0)$ has a unique eigenvalue of multiplicity $d-1$, lying below the essential spectrum.*

• *If $(\gamma,\lambda) \in \overline{\mathcal{C}_0}$, then the operator $H^s_{\gamma\lambda}(0)$ has no eigenvalues below the essential spectrum.*

• *If $(\gamma,\lambda) \in \mathcal{C}_1 \cup \partial\mathcal{C}_2$, then the operator $H^s_{\gamma\lambda}(0)$ has a single simple eigenvalue lying below the essential spectrum.*

• *If $(\gamma,\lambda) \in \mathcal{C}_2$, then the operator $H^s_{\gamma\lambda}(0)$ has exactly two eigenvalues lying below the essential spectrum.*

Let us define the boundaries for the connected components (Fig. 2)

$$\partial\mathcal{S}_0 := \left\{(\gamma,\lambda) \in \mathbb{R}^2 \colon \lambda = \lambda_0\right\},$$

$$\partial\mathcal{C}_0 := \left\{(\gamma,\lambda) \in \mathbb{R}^2 \colon 1 + \gamma a(0) + \lambda\left(d + \frac{\gamma}{2}\right)b(0) = 0,\ \lambda > -2\frac{a(0)}{b(0)}\right\},$$

$$\partial\mathcal{C}_2 := \left\{(\gamma,\lambda) \in \mathbb{R}^2 \colon 1 + \gamma a(0) + \lambda\left(d + \frac{\gamma}{2}\right)b(0) = 0,\ \lambda < -2\frac{a(0)}{b(0)}\right\},$$

$$P := \partial\mathcal{C}_2 \cap \partial\mathcal{S}_0.$$

Definition. Let f be a solution of the equation $H_{\gamma\lambda}(0)f = \mathcal{E}_{\min}(0)f$.

• If $f \in L^{2,e}(\mathbb{T}^d)$, then $\mathcal{E}_{\min}(0)$ is called the lower threshold eigenvalue $H_{\gamma\lambda}(0)$.

• If $f \in L^{1,e}(\mathbb{T}^d)\backslash L^{2,e}(\mathbb{T}^d)$, then $\mathcal{E}_{\min}(0)$ is called the lower threshold resonance $H_{\gamma\lambda}(0)$.

Existence of a threshold eigenvalue and a threshold resonance of the operator $H_{\gamma\lambda}(0)$ at the boundaries of connected components $\partial\mathcal{S}_0$ and $\partial\mathcal{C}_j$, $j = 0,2$, proves

Theorem 2. *The following statements are true.*

• *Let $d \geqslant 3$. If $(\gamma,\lambda) \in \partial\mathcal{S}_0 \backslash P$, then $\mathcal{E}_{\min}(0)$ is the lower threshold eigenvalue of multiplicity $d-1$ of the operator $H_{\gamma\lambda}(0)$.*

• *Let $d = 3,4$. If $(\gamma,\lambda) \in \partial\mathcal{C}_j \backslash P$, $j = 0,2$, then $\mathcal{E}_{\min}(0)$ is the lower threshold resonance of the operator $H_{\gamma\lambda}(0)$.*

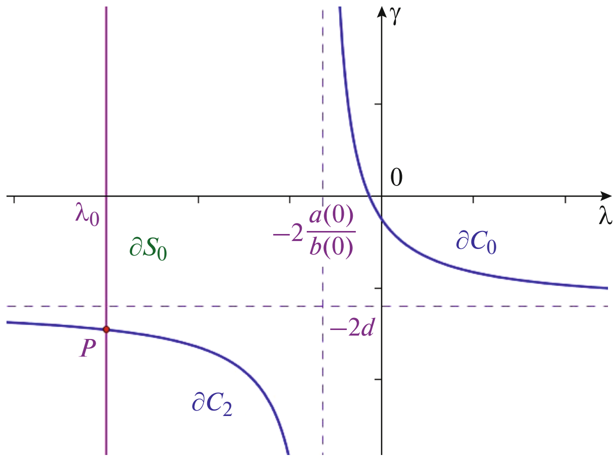

Fig. 2. Boundaries of connected components.

- *Let* $d \geqslant 5$. *If* $(\gamma,\lambda) \in \partial\mathcal{C}_j \backslash P$, $j = 0,2$, *then* $\mathcal{E}_{\min}(0)$ *is the lower threshold eigenvalue of the operator* $H_{\gamma\lambda}(0)$.
- *Let* $d = 3,4$. *If* $(\gamma,\lambda) \in P$, *then* $\mathcal{E}_{\min}(0)$ *is the lower threshold resonance and the lower threshold eigenvalue of multiplicity* $d-1$ *of the operator* $H_{\gamma\lambda}(0)$.
- *Let* $d \geqslant 5$. *If* $(\gamma,\lambda) \in P$, *then* $\mathcal{E}_{\min}(0)$ *is the lower threshold eigenvalue of multiplicity* d *of the operator* $H_{\gamma\lambda}(0)$.

3. PROOF OF THE MAIN RESULTS

Let us define symmetric and antisymmetric subspaces $L^{2,es}(\mathbb{T}^d) \subset L^{2,e}(\mathbb{T}^d)$ and $L^{2,ea}(\mathbb{T}^d) \subset L^{2,e}(\mathbb{T}^d)$, respectively, as

$$L^{2,es}(\mathbb{T}^d) := \left\{ f \in L^{2,e}(\mathbb{T}^d)\colon f(p_1,\ldots,p_d) = \Pi_2 f(p_1,\ldots,p_d),\ p_1,\ldots,p_d \in \mathbb{T} \right\},$$

$$L^{2,ea}(\mathbb{T}^d) := \left\{ f \in L^{2,e}(\mathbb{T}^d)\colon f(p_1,\ldots,p_d) = -\Pi_2 f(p_1,\ldots,p_d),\ p_1,\ldots,p_d \in \mathbb{T} \right\},$$

where Π_2 denotes the permutation of any two variables p_i and p_j, $i,j = 1,\ldots,d$. The subspaces $L^{2,es}(\mathbb{T}^d)$ and $L^{2,ea}(\mathbb{T}^d)$ are invariant with respect to $H_{\gamma\lambda}(0)$.

By $H^s_{\gamma\lambda}(0)$ and $H^a_{\lambda}(0)$ we denote the contractions of $H_{\gamma\lambda}(0)\big|_{L^{2,es}(\mathbb{T}^d)}$ and $H_{\gamma\lambda}(0)\big|_{L^{2,ea}(\mathbb{T}^d)}$ of the operator $H_{\gamma\lambda}(0)$ on $L^{2,es}(\mathbb{T}^d)$ and $L^{2,ea}(\mathbb{T}^d)$, respectively. The operators $H^s_{\gamma\lambda}(0)$ and $H^a_{\lambda}(0)$ operate in $L^{2,es}(\mathbb{T}^d)$ and $L^{2,ea}(\mathbb{T}^d)$ by formulas

$$H^s_{\gamma\lambda}(0) := H_0(0) + V^s_{\gamma\lambda} \quad \text{and} \quad H^a_{\lambda}(0) := H_0(0) + V^a_{\lambda},$$

where

$$V^s_{\gamma\lambda} f(p) = \frac{\gamma}{(2\pi)^d} \int_{\mathbb{T}^d} f(q)\mathrm{d}q + \frac{\lambda}{(2\pi)^d} \frac{1}{2(d-1)} \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} (\cos p_i + \cos p_j) \int_{\mathbb{T}^d} (\cos q_i + \cos q_j) f(q)\mathrm{d}q,$$

$$V^a_{\lambda} f(p) = \frac{\lambda}{(2\pi)^d} \frac{1}{2(d-1)} \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} (\cos p_i - \cos p_j) \int_{\mathbb{T}^d} (\cos q_i - \cos q_j) f(q)\mathrm{d}q$$

are integral operators. Thus, the equality holds:

$$\sigma(H_{\gamma\lambda}(0)) = \sigma(H^s_{\gamma\lambda}(0)) \cup \sigma(H^a_{\lambda}(0)).$$

Let us consider the equation for the eigenfunctions of the operators $H^s_{\gamma\lambda}(0)$ and $H^a_{\lambda}(0)$. To do this, it is enough to solve the equations

$$H^s_{\gamma\lambda}(0)f = zf, \quad H^a_{\lambda}(0)f = zf$$

in $z \in \mathbb{C}\backslash[0,4d]$ for nonzero f. Here, we apply the theory of Fredholm determinants (see, e.g., [32]).

Let us write down the Fredholm determinants corresponding to the operators $H^s_{\gamma\lambda}(0)$ and $H^a_{\lambda}(0)$, respectively:

$$\Delta^{(s)}_{\gamma\lambda}(z) := \Delta_{\gamma 0}(z)\Delta_{0\lambda}(z) - d\gamma\lambda b(z)^2, \tag{1}$$

where

$$\Delta_{\gamma 0}(z) := 1 + \gamma a(z), \qquad \Delta_{0\lambda}(z) := 1 + \lambda(c(z) + (d-1)e(z)),$$

$$\Delta^{(a)}_{\lambda}(z) := [1 + \lambda(c(z) - e(z))]^{d-1}.$$

According to ([33], Lemma 3.3; [34], Lemma 4.1), we obtain the following statements.

Lemma 2. *The function $a(\cdot)$ is real-valued on $\mathbb{R}\backslash[0,4d]$ and has the following asymptotics for $d \geq 3$:*

$$a(z) = a(0) + L_d(z), \ \ as \ \ z \nearrow 0, \tag{2}$$

where

$$L_d(z) = \begin{cases} \dfrac{1}{2}\dfrac{2^{3d/2}}{(d-2)!!}\left(\dfrac{\pi}{2}\right)^{\left[\frac{d}{2}\right]}(-z)^m \ln(-z) + O(-z)^{m+1}, & d = 2(m+1); \\ \dfrac{2^{3d/2}}{(d-2)!!}\left(\dfrac{\pi}{2}\right)^{\left[\frac{d}{2}\right]}\dfrac{\pi}{2\sqrt{-z}}(-z)^m + O(-z)^{m+1}, & d = 2m+1, \ m = 0,1,2,\ldots. \end{cases} \tag{3}$$

Lemma 3. *The function $\Delta^{(s)}_{\gamma\lambda}(z)$ is real-valued on $\mathbb{R}\backslash[0,4d]$ and has the following asymptotics*:

$$\Delta^{(s)}_{\gamma\lambda}(z) = C_0(\gamma,\lambda) + C_1(\gamma,\lambda)L_d(z), \ \ as \ \ z \nearrow 0, \tag{4}$$

where

$$C_0(\gamma,\lambda) = \frac{4d - \lambda(2d+\gamma) + (4d\gamma + 2d\lambda(2d+\gamma))a(0)}{4d},$$

$$C_1(\gamma,\lambda) = \frac{4d\gamma + 2d\lambda(2d+\gamma)}{4d},$$

the function $L_d(z)$ is determined by equality (3).

Proof. It is easy to prove that the equalities hold

$$c(z) + (d-1)e(z) = \frac{2d - z}{2}b(z), \tag{5}$$

$$a(z) - b(z) = \frac{1}{2d}(1 + za(z)). \tag{6}$$

Hence,

$$b(z) = \frac{1}{2d}((2d - z)a(z) - 1).$$

Using (5) and (6), we represent (1) in the form

$$\Delta^{(s)}_{\gamma\lambda}(z) = \frac{4d - \lambda(2d - z + \gamma)}{4d} + \frac{4d\gamma + \lambda(2d - z)(2d - z + \gamma)}{4d}a(z), \qquad d \geqslant 3. \tag{7}$$

Using equality (2) for (7), we obtain (4).

□

Asymptotics (4) of the Fredholm determinant $\Delta^{(s)}_{\gamma\lambda}(z)$ shows that the number of eigenvalues of the operator $H^s_{\gamma\lambda}(0)$ lying below its essential spectrum remains constant at each of the boundaries of the connected components $\mathcal{C}_j$, $j=0,2$. Because the first and second terms of the asymptotics of the Fredholm determinant $\Delta^{(s)}_{\gamma\lambda}(z)$ are not equal to zero simultaneously.

Lemma 4. *The following statements are true.*

- *If $(\gamma,\lambda)\in\partial\mathcal{S}_0$, then $\mathcal{E}_{\min}(0)$ is the lower threshold eigenvalue of multiplicity $d-1$ of the operator $H^a_\lambda(0)$.*
- *Let $d=3,4$. If $(\gamma,\lambda)\in\partial\mathcal{C}_j$, $j=0,2$, then $\mathcal{E}_{\min}(0)$ is the lower threshold resonance of the operator $H^s_{\gamma\lambda}(0)$.*
- *Let $d\geqslant 5$. If $(\gamma,\lambda)\in\partial\mathcal{C}_j$, $j=0,2$, then $\mathcal{E}_{\min}(0)$ is the lower threshold eigenvalue of the operator $H^s_{\gamma\lambda}(0)$.*

Proof. (i) Let f be the solution to the equation

$$H^a_\lambda(0)f=\mathcal{E}_{\min}(0)f.$$

Then we obtain the validity of the equalities

$$\mathcal{E}_0(p)f(p)+\frac{\lambda}{(2\pi)^d}\frac{1}{2(d-1)}\sum_{i=1}^{d-1}\sum_{j=i+1}^{d}C(\cos p_i-\cos p_j)=0,$$

$$f(p)=-\frac{\lambda}{(2\pi)^d}\frac{1}{2(d-1)}\frac{\sum_{i=1}^{d-1}\sum_{j=i+1}^{d}C(\cos p_i-\cos p_j)}{\mathcal{E}_0(p)},$$

where $C=\int_{\mathbb{T}^d}(\cos q_i-\cos q_j)f(q)\mathrm{d}q$.

The function $f(\cdot)$ consists of a linear combination of the functions $f_i(\cdot)$. For example, for $d=3$

$$f_1(p)=\frac{\cos p_1-\cos p_2}{\mathcal{E}_0(p)},\quad f_2(p)=\frac{2\cos p_3-\cos p_2-\cos p_1}{\mathcal{E}_0(p)}.$$

For $d=4$,

$$f_1(p)=\frac{\cos p_1-\cos p_2}{\mathcal{E}_0(p)},\quad f_2(p)=\frac{2\cos p_3-\cos p_2-\cos p_1}{\mathcal{E}_0(p)},$$

$$f_3(p)=\frac{\cos p_1+\cos p_2+\cos p_3-3\cos p_4}{\mathcal{E}_0(p)}.$$

Because $|\mathcal{E}_0(p)|=O(p^2)$ and $|\cos p_i-\cos p_j|=O(p^2)$ as $|p|\to 0$, we get $f\in L^{2,e}(\mathbb{T}^d)$. Then, by Definition 2, it follows that the lower point $z=\mathcal{E}_{\min}(0)=0$ of the essential spectrum is the threshold eigenvalue of multiplicity $d-1$ of the operator $H^a_\lambda(0)$.

(ii) Let f be a solution to the equation

$$H^s_{\gamma\lambda}(0)f=\mathcal{E}_{\min}(0)f.$$

Then the equality holds

$$\mathcal{E}_0(p)f(p)+\frac{\gamma}{(2\pi)^d}C_1+\frac{\lambda}{(2\pi)^d}\frac{1}{(d-1)}\sum_{i=1}^{d-1}\sum_{j=i+1}^{d}C_2(\cos p_i+\cos p_j)=0,$$

where $C_1=\int_{\mathbb{T}^d}f(q)\mathrm{d}q$, $C_2=\int_{\mathbb{T}^d}\cos q_i f(q)\mathrm{d}q$.

Let us represent the function f as a sum

$$f(p)=f_1(p)+f_2(p),$$

where

$$f_1(p) = -\frac{\gamma}{(2\pi)^d}\frac{C_1}{\mathcal{E}_0(p)}, \quad f_2(p) = -\frac{\lambda}{(2\pi)^d}\frac{1}{(d-1)}\frac{C_2\sum_{i=1}^{d-1}\sum_{j=i+1}^{d}(\cos p_i + \cos p_j)}{\mathcal{E}_0(p)}.$$

Let us show that $f_1, f_2 \notin L^{2,e}(\mathbb{T}^d)$ or $f_1, f_2 \in L^{2,e}(\mathbb{T}^d)$:

$$\int_{\mathbb{T}^d} f_1^2(p)\mathrm{d}p = \frac{\gamma^2 C_1^2}{(2\pi)^{2d}}\int_{\mathbb{T}^d}\frac{\mathrm{d}p}{\mathcal{E}_0^2(p)} = \frac{\gamma^2 C_1^2}{(2\pi)^{2d}} I_1 + \frac{\gamma^2 C_1^2}{(2\pi)^{2d}} I_2,$$

$$I_1 = \int_{U_\delta(0)}\frac{\mathrm{d}p}{\mathcal{E}_0^2(p)}, \quad I_2 = \int_{\mathbb{T}^d\setminus U_\delta(0)}\frac{\mathrm{d}p}{\mathcal{E}_0^2(p)}.$$

Note that I_2 is analytic at a point $p = \theta$. Using the arguments of Lemma 2, the integral I_1 can be represented as

$$I_1 = 2^{\frac{d}{2}}\left(R_0\int_0^\eta \frac{r^{d-1}}{r^4}\mathrm{d}r + R_2\int_0^\eta \frac{r^{d+1}}{r^4}\mathrm{d}r + R_4\int_0^\eta \frac{r^{d+3}}{r^4}\mathrm{d}r\right).$$

Let $d = 3, 4$. Then the integral I_1 is infinite. Hence, $f_1 \notin L^{2,e}(\mathbb{T}^d)$. In addition, for the function f_2 using the same relations as above, we obtain $f_2 \notin L^{2,e}(\mathbb{T}^d)$. From here, by virtue of Definition 2, it follows that the lower point $z = \mathcal{E}_{\min}(0) = 0$ of the essential spectrum is the threshold resonance of the operator $H^s_{\gamma\lambda}(0)$.

(iii) Let $d \geqslant 5$. Then the integral I_1 is finite. Hence, $f_1, f_2 \in L^{2,e}(\mathbb{T}^d)$. Therefore, by Definition 2, it follows that the lower point $z = \mathcal{E}_{\min}(0) = 0$ of the essential spectrum is the threshold eigenvalue of the operator $H^s_{\gamma\lambda}(0)$.

□

Theorem 2 follows immediately from the proof of Lemma 4.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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