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Charged Zipoy–Voorhees metric in string theory

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ABSTRACT

We have presented a solution for a charged Zipoy–Voorhees metric within the framework of low-energy effective field theory for heterotic string theory. This exact solution represents a deformed gravitational field in four-dimensional spacetime, characterized by three key parameters: the mass M, the electric charge Q, and the deformation parameter γ, which controls deviations from spherical symmetry. Unlike standard black hole solutions, this metric describes a naked singularity, meaning it lacks an event horizon and exposes the central singularity to external observers. To examine the physical properties of this spacetime, we analyzed the geodesic motion of test particles and photons. The study of geodesic trajectories provides insight into the effects of the deformation parameter γ and charged parameter $b = Q^2/2M$ on orbital motion, shadow, and potential astrophysical observables. Finally, a constraint on charge and deformation parameter using observed shadow radius of M87* has been obtained.

1. Introduction

String theory is broadly acknowledged as a consistent theoretical model that successfully integrates quantum mechanics with general relativity, potentially elucidating gravity at the quantum scale. Consequently, research into the low-energy regimes of string theory, which account for gravitational effects, continues to be a key area of interest in gravitational physics. In heterotic string theory, the low-energy regime includes essential fields like the graviton, an Abelian gauge field, the dilaton, and second-rank antisymmetric tensor fields. This regime leads to a distinctive black hole solution called the Kerr–Sen spacetime [1], obtained through the Hassan–Sen transformation [2]. Employing this transformation, studies such as [3–5] have explored a series of charged black hole solutions within the low-energy framework of heterotic string theory.

The solution under consideration is the Zipoy–Voorhees metric [6,7], also known as the γ-metric [8] or the q-metric [9–13], which is part of the broader class of Weyl solutions [14–16]. The influence of the scalar field within the γ-spacetime framework has been explored in Ref. [17]. It is important to highlight that these spacetime metrics fall under Weyl's class of solutions, which describe a wide range of static, axisymmetric vacuum solutions in general relativity.

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The structure of the Zipoy–Voorhees (γ-metric) solution was extensively analyzed in Ref. [18], while various interior solutions corresponding to the exterior metric have been discussed in the literature [19–21]. In recent years, numerous studies have explored the observational properties of the γ-metric, shedding light on its astrophysical implications.

For instance, its optical appearance was investigated in [22,23], while the characteristics of thin accretion disks surrounding the γ-metric were studied in [24,25]. The impact of the deformation parameter γ on shadows and light bending effects has been explored in [26,27]. Additionally, studies on geodesic motion in this spacetime are found in [11,28,29], and the motion of spinning particles was examined in [30]. The behavior of test particles undergoing harmonic oscillations in the γ-metric was addressed in [31], and the phenomenon of neutrino lensing was analyzed in [32,33].

Moreover, particle dynamics in the γ-spacetime in the presence of a scalar field were investigated in [17], whereas the motion of charged particles in an external magnetic field was studied in [34,35]. The collision of massive particles near the singularity of this spacetime was examined in [36]. Additionally, connections between the γ-metric and other exotic spacetimes, such as wormholes, were discussed in [37]. The effect of the scalar field in particle dynamics has been studied in [38–46].

The key question that arises is whether an analytical solution for the charged Zipoy–Voorhees spacetime can be obtained. In the framework of general relativity the solution for the charged γ-metric has been presented in Ref. [47] and quantum wave packets obeying Klein–Gordon equation are used to probe timelike naked singularities and shown that by rigorous mathematical calculations that the outermost singularity developed in the charged and uncharged Zipoy–Voorhees spacetime on the equatorial plane is quantum mechanically singular for all values of the deformation parameter γ in [48]. However, the charged Zipoy–Voorhees solution presented in Ref. [47] does not satisfy the Einstein–Maxwell field equations. Therefore, in this paper, we aim to derive an exact solution for the charged Zipoy–Voorhees spacetime within the framework of string theory by employing the Hassan–Sen transformation.

The Hassan–Sen transformation is a highly valuable technique for deriving charged black hole solutions from neutral vacuum solutions in general relativity [2]. By selecting an appropriate seed metric, this method enables the construction of solutions with distinctive characteristics, such as electric and magnetic charges, rotational properties, and nontrivial couplings with the dilaton field. A notable example is its application to the Kerr metric, which results in a charged, rotating black hole solution in heterotic string theory, effectively generalizing the Kerr–Newman solution of Einstein–Maxwell theory.

As described in Ref. [1], when a given metric $\tilde{g}_{\mu\nu}$ serves as a vacuum solution to Einstein's equations ($R_{\mu\nu} = 0$), the Hassan–Sen transformation provides a systematic method for generating new solutions within the low-energy regime of heterotic string theory. This transformation extends the original metric by incorporating additional fields that interact with gravity, thereby modifying the seed solution through contributions from the antisymmetric Kalb–Ramond field $B_{\mu\nu}$, the Maxwell field A_μ, the dilaton field Φ, and the string-frame metric tensor $G_{\mu\nu}$. The resulting configuration satisfies the field equations derived from the effective action of low-energy heterotic string theory, leading to the transformation

$$\tilde{g}_{\mu\nu} \rightarrow \{G_{\mu\nu}, B_{\mu\nu}, A_\mu, \Phi\} \ . \tag{1}$$

In this framework, the string-frame metric $G_{\mu\nu}$ is obtained through a conformal rescaling of the original seed metric, with the scaling factor determined by the dilaton field Φ. The Maxwell field A_μ arises from the interplay between the seed metric and transformation parameters, often associated with underlying hidden symmetries of the system. The antisymmetric Kalb–Ramond field $B_{\mu\nu}$, which plays a crucial role in string interactions, is constructed using the seed metric and additional transformation parameters. Meanwhile, the dilaton field Φ, which governs the strength of string interactions, is adjusted to maintain consistency with the equations of motion of heterotic string theory. Through this approach, the Hassan–Sen transformation enables the construction of new, physically meaningful solutions that extend Einstein's general relativity within a string-theoretic framework.

The paper is organized as follows. In Section 2, we provide necessary equations for tensor, vector, and scalar fields. We also provide a guideline on how to obtain a charged black hole solution in string theory, given a vacuum solution in general relativity. In Section 3, we demonstrate the derivation of the Sen metric, also known as the Gibbons–Maeda–Garfinkle–Horowitz–Strominger (GMGHS) metric, starting from the Schwarzschild spacetime. In Section 4, we construct the explicit form of the charged Zipoy–Voorhees spacetime by applying the Hassan–Sen transformation. In Section 5, we test charged Zipoy–Voorhees spacetime by considering motion of neutral and charged particle. Finally, in Section 6, we summarize obtained results.

Throughout the paper, we choose the $(-,+,+,+)$ signature for the metric tensor and spherical coordinates $x^\alpha = (t, r, \theta, \phi)$. Greek (Latin) indices run from 0 to 3 (1 to 3).

2. Field equations

Following the application of the Hassan–Sen transformation, the action for the system is given by [1]:

$$S = \int d^4x\sqrt{-G}e^{-\Phi}\left[-R + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{8}F_{\mu\nu}F^{\mu\nu}\right] , \tag{2}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ represents the Faraday tensor associated with the Maxwell field A_μ, and $H_{\mu\nu\rho}$ is defined as:

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu} - \frac{1}{4}\left(A_\mu F_{\nu\rho} + A_\rho F_{\mu\nu} + A_\nu F_{\rho\mu}\right) \ . \tag{3}$$

Varying the action (2) yields the following equations of motion:

$$R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R + \nabla_\mu\nabla_\nu\Phi - G_{\mu\nu}\left(\nabla^2\Phi - \frac{1}{2}\nabla_\rho\Phi\nabla^\rho\Phi\right) = \frac{1}{4}\left(F_{\mu\rho}F^\rho_\nu - \frac{1}{4}G_{\mu\nu}F + H_{\mu\alpha\beta}H^{\alpha\beta}_\nu - \frac{1}{6}G_{\mu\nu}H\right) , \tag{4}$$

$$\nabla_\mu H^{\mu\nu\rho} = H^{\mu\nu\rho}\nabla_\mu \Phi \ , \tag{5}$$

$$\nabla_\mu F^{\mu\nu} = F^{\mu\nu}\nabla_\mu \Phi + \frac{1}{2}F^{\alpha\beta}H_{\nu\alpha\beta} \ , \tag{6}$$

$$\nabla_\mu \Phi \nabla^\mu \Phi - 2\nabla^2 \Phi = R - \frac{1}{8}F - \frac{1}{12}H \ . \tag{7}$$

where $H = H_{\mu\nu\rho}H^{\mu\nu\rho}$ and $F = F_{\mu\nu}F^{\mu\nu}$. To obtain exact solutions to these field Eqs. (4)–(7) in the presence of charge, we start with an axially symmetric vacuum solution of Einstein's equations in general relativity, represented by the metric [3]:

$$d\tilde{s}^2 = \tilde{g}_{tt}dt^2 + 2\tilde{g}_{t\phi}dtd\phi + \tilde{g}_{\phi\phi}d\phi^2 + \tilde{g}_{rr}dr^2 + \tilde{g}_{\theta\theta}d\theta^2 \ . \tag{8}$$

The corresponding string-frame solutions for the tensor, vector, and dilaton fields are then expressed as:

$$d\tilde{s}'^2 = \frac{\tilde{g}_{tt}}{\Lambda^2}\left[dt + \frac{\tilde{g}_{t\phi}}{\tilde{g}_{tt}}(1+s^2)d\phi\right]^2 + \tilde{g}_{rr}dr^2 + \tilde{g}_{\theta\theta}d\theta^2 + \left(\tilde{g}_{\phi\phi} - \frac{\tilde{g}_{t\phi}^2}{\tilde{g}_{tt}}\right)d\phi^2 \ , \tag{9}$$

$$B_{t\phi} = -B_{\phi t} = \frac{\tilde{g}_{t\phi}}{\Lambda}s^2 \ , \tag{10}$$

$$A = \frac{s\sqrt{1+s^2}}{\Lambda}\left[(1+\tilde{g}_{tt})dt + \tilde{g}_{t\phi}d\phi\right] \ , \tag{11}$$

$$\Phi = -\ln \Lambda, \tag{12}$$

where Λ is defined as:

$$\Lambda = 1 + s^2(1+\tilde{g}_{tt}) \ , \tag{13}$$

and s is a constant related to the charge of the central object. The string-frame solutions fully satisfy the field Eqs. (4)–(7). Notably, the transformation impacts the total mass of the central object in general relativity by introducing the charge parameter s.

To move to the Einstein frame, a conformal transformation of the metric is performed [1]:

$$g_{\mu\nu} = e^{-\Phi}\tilde{g}'_{\mu\nu} \ . \tag{14}$$

After this transformation, the action in the Einstein frame becomes

$$S = \int d^4x\sqrt{-g}\left(-R + \frac{1}{12}e^{-2\Phi}H_{\mu\nu\rho}H^{\mu\nu\rho} - g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{8}e^{-\Phi}F_{\mu\nu}F^{\mu\nu}\right) \ . \tag{15}$$

By varying the action with respect to $g_{\mu\nu}$, Φ, $B_{\mu\nu}$, and A_μ The corresponding field equations are obtained as

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{4}e^{-\Phi}\left(F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) + \frac{1}{4}e^{-2\Phi}\left(H_{\mu\lambda\sigma}H_\nu{}^{\lambda\sigma} - \frac{1}{6}g_{\mu\nu}H_{\rho\sigma\delta}H^{\rho\sigma\delta}\right) \ , \tag{16}$$

$$\nabla^\mu\nabla_\mu\Phi = \frac{1}{6}e^{-2\Phi}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{8}e^{-\Phi}F_{\mu\nu}F^{\mu\nu} \ , \tag{17}$$

$$\nabla_\lambda\left(e^{-2\Phi}H^{\lambda\mu\nu}\right) = 0 \ , \tag{18}$$

$$\nabla_\nu\left(e^{-\Phi}F^{\nu\mu}\right) + \frac{1}{6}e^{-2\Phi}H^{\mu\nu\rho}F_{\nu\rho} = 0 \ . \tag{19}$$

Thus, in the Einstein frame, the charged black hole in string theory takes the form [3]:

$$ds^2 = \frac{\tilde{g}_{tt}}{\Lambda}\left[dt + \frac{\tilde{g}_{t\phi}}{\tilde{g}_{tt}}(1+s^2)d\phi\right]^2 + \Lambda\left[\tilde{g}_{rr}dr^2 + \tilde{g}_{\theta\theta}d\theta^2 + \left(\tilde{g}_{\phi\phi} - \frac{\tilde{g}_{t\phi}^2}{\tilde{g}_{tt}}\right)d\phi^2\right] \ . \tag{20}$$

This transformed metric encapsulates the effects of charge and additional fields, distinguishing it from the original vacuum solution.

Beyond generating specific solutions, the Hassan–Sen transformation unveils deeper insights into the structure of heterotic string theory. It leverages hidden symmetries, often linked to the theory's duality properties, to connect classical general relativity solutions to string-theoretic frameworks. As a symmetry of the low-energy effective action, the transformation facilitates the creation of a diverse array of solutions, expanding the repertoire of black hole and cosmological configurations in string theory. For instance, starting with simple vacuum solutions like the Schwarzschild metric, the transformation can introduce charge and dilaton effects, leading to the Kerr–Sen spacetime or other exotic geometries.

The method's versatility extends to exploring astrophysical implications. The charged solutions derived from the Hassan–Sen transformation can model phenomena such as charged rotating compact objects, offering a string-theoretic perspective on objects traditionally described by Einstein–Maxwell theory. Furthermore, the inclusion of the Kalb–Ramond field $B_{\mu\nu}$ introduces a topological richness to the spacetime, potentially influencing geodesic motion and gravitational interactions in ways not captured by general relativity alone.

In practical terms, the transformation process begins with a neutral seed metric, such as the Curzon–Chazy or Kerr metric, and systematically builds a charged counterpart by defining the fields $G_{\mu\nu}$, $B_{\mu\nu}$, A_μ, and Φ in terms of transformation parameters like s. The resulting solutions are then tested against the field Eqs. (4)–(7) to ensure physical consistency. This approach not only bridges general relativity and string theory but also provides a framework for studying the interplay of gravity, electromagnetism, and scalar fields in a unified context.

Ultimately, the Hassan–Sen transformation stands as a powerful tool for probing the landscape of string-inspired gravitational backgrounds. By transforming vacuum solutions of Einstein's equations into richly structured spacetimes, it enhances our understanding of the connections between classical gravity and the quantum realms of string theory, paving the way for new theoretical and potentially observable predictions.

3. Sen/Gibbons–Maeda–Garfinkle–Horowitz–Strominger spacetime

It is well known that in the framework of string theory the Sen metric or GMGHS metric can be expressed as [49,50]:

$$ds_0^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{\left(1-\frac{2M}{r}\right)}dr^2 + r\left(r-\frac{Q^2}{M}\right)d\Omega\ , \tag{21}$$

where $d\Omega = d\theta^2 + \sin^2\theta d\phi^2$ denotes the standard metric on the two-sphere, M is a mass and Q is a charge of a black hole.

Here, we demonstrate the procedure for obtaining the how to get Sen/GMGHS metric, from the Schwarzschild metric by applying the Hassan–Sen transformation. The Schwarzschild spacetime is described by the following line element:

$$d\tilde{s}^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2m}{r}\right)} + r^2 d\Omega\ , \tag{22}$$

where m represents the mass of the black hole. By applying the Hassan–Sen transformation, we systematically incorporate additional fields — including the dilaton, the Maxwell field, and the Kalb–Ramond field — into this vacuum solution, thereby generating the charged black hole metric within the framework of heterotic string theory yields

$$d\tilde{s}'^2 = -\frac{\left(1-\frac{2m}{r}\right)}{\Lambda^2}dt^2 + \frac{1}{\left(1-\frac{2m}{r}\right)}dr^2 + r^2 d\Omega\ , \tag{23}$$

$$B_{t\phi} = -B_{\phi t} = 0\ , \tag{24}$$

$$A_t = \frac{s\sqrt{1+s^2}}{\Lambda}(1+\tilde{g}_{tt})\ , \tag{25}$$

$$\Phi = -\ln\Lambda, \tag{26}$$

where function Λ in the Schwarzschild spacetime is defined as

$$\Lambda = 1 + s^2(1+\tilde{g}_{tt}) = 1 + \frac{2ms^2}{r}\ . \tag{27}$$

After applying the following conformal transformation $ds^2 = e^{-\Phi}d\tilde{s}'^2 = \Lambda d\tilde{s}'^2$, the charged black hole solution in the Einstein frame can be expressed as

$$ds^2 = -\frac{r-2m}{r+2ms^2}dt^2 + \frac{r+2ms^2}{r-2m}dr^2 + r(r+2ms^2)d\Omega\ . \tag{28}$$

Notice that the total mass and charge of the black hole are introduced as follows:

$$m = \frac{2M^2-Q^2}{2M}\ , \qquad s^2 = \frac{Q^2}{2M^2-Q^2}\ . \tag{29}$$

Then the metric (28) can be rewritten as

$$ds^2 = -\left(1-\frac{2M}{r+\frac{Q^2}{M}}\right)dt^2 + \frac{1}{\left(1-\frac{2M}{r+\frac{Q^2}{M}}\right)}dr^2 + r\left(r+\frac{Q^2}{M}\right)d\Omega\ ,$$

which is known as the Sen metric [1]. Hereafter, introducing a new radial coordinate $r \to r - Q^2/M$, the Sen metric reduces to Gibbons–Maeda–Garfinkle–Horowitz–Strominger metric as shown in Eq. (21), while the associated scalar, vector and tensor fields are given as

$$\Phi = -\ln\left(1-\frac{Q^2}{Mr}\right)\ , \quad A_\mu = \frac{Q}{r}\delta_{\mu t}\ , \quad B_{\mu\nu} = 0\ . \tag{30}$$

It is evident that the event horizon of the GMGHS black hole remains identical to that of the Schwarzschild black hole. The black hole horizon is determined as $r_h = 2M$, which is simply obtained from equation $g_{tt} = 0$. In the GMGHS spacetime (21), the curvature invariants, namely, the Ricci scalar and the Kretschmann scalar can be easily presented as

$$\begin{aligned} R &= \frac{Q^4(r-2M)}{2r^3\left(Mr-Q^2\right)^2}\ , \\ K &= \frac{48M^6}{r^2\left(Mr-Q^2\right)^4}\left[1-\frac{3Q^2}{Mr}-\frac{Q^4(2r-45M)}{12M^3r^2}+\frac{Q^6(5r-54M)}{24M^4r^3}+\frac{Q^8\left(3r^2-20Mr+108M^2\right)}{192M^6r^4}\right]\ . \end{aligned} \tag{31}$$

which contain two singularities: the first singularity is located at the origin, $r = 0$, while the second one appears at $r = Q^2/M$. An interesting observation is that when the charge and mass satisfy the condition $Q^2 = 2M^2$, the singularity coincides with the black hole horizon, leading to a special limiting case of the solution. In this case, the metric (21) reduces to the solution for a naked singularity.

Here we study basic thermodynamic quantities of the spherical symmetric static black hole in the string. The Hawking temperature over the surface of the black hole in the string is determined as

$$T = \frac{1}{4\pi}\frac{dg_{tt}}{dr}\Big|_{r=r_h} = \frac{1}{8\pi M}\ , \tag{32}$$

which is the same as that of the Schwarzschild black hole. However, the entropy of the black hole is proportional to area of the horizon, i.e. $S \sim A$ and can be found as

$$S = \frac{1}{4}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sqrt{g_{\theta\theta}g_{\phi\phi}}\Big|_{r=r_h} = 4\pi M^2\left(1 - \frac{Q^2}{2M^2}\right)\ . \tag{33}$$

which is different from that is presented in the Schwarzschild spacetime. Interestingly, the entropy of GMGHS black hole vanishes when $Q^2 = 2M^2$.

The Gibbons–Maeda–Garfinkle–Horowitz–Strominger (GMGHS) spacetime is a solution in low-energy heterotic string theory, describing a static, spherically symmetric, charged black hole coupled to a dilaton field. Independently discovered by Gibbons and Maeda, and later by Garfinkle, Horowitz, and Strominger, it extends classical general relativity by incorporating string-theoretic corrections, notably the dilaton, a scalar field that influences the gravitational and electromagnetic interactions. A curvature singularity exists at $r = Q^2/M$. For $Q^2 < 2M^2$, this lies inside the horizon, shielded by it. If $Q^2 = 2M^2$, the singularity coincides with the horizon at $r = 2M$, and for $Q^2 > 2M^2$, it becomes a naked singularity as the horizon disappears. When $Q^2 = 2M^2$, the horizon and singularity align, marking a critical transition. Unlike extremal Reissner–Nordström black holes, this configuration lacks a smooth near-horizon geometry like $AdS_2 \times S^2$ without additional regularization (e.g., dilaton potential). In the extremal case ($Q^2 = 2M^2$), the alignment of the singularity with the horizon suggests a breakdown of the classical black hole picture, potentially indicating a naked singularity or a need for higher-dimensional resolution, as explored in string theory. The dilaton modifies the spacetime geometry, shrinking the spherical surfaces' effective radius to $r(r - Q^2/M)$, impacting phenomena like accretion disks and geodesics.

4. Charged Zipoy-Voorhees metric

In this case section, we will show how to get the charged Zipoy–Voorhees metric in the low-energy effective field theory for heterotic string theory. The components of the metric tensor in the Zipoy–Voorhees spacetime can be written as [23,30]

$$\begin{aligned}
\tilde{g}_{tt} &= -\left(1 - \frac{2m}{\gamma r}\right)^{\gamma}\ ,\\
\tilde{g}_{rr} &= \left(1 - \frac{2m}{\gamma r}\right)^{\gamma^2-\gamma-1}\left(1 - \frac{2m}{\gamma r} + \frac{m^2\sin^2\theta}{\gamma^2 r^2}\right)^{1-\gamma^2}\ ,\\
\tilde{g}_{\theta\theta} &= \left(1 - \frac{2m}{\gamma r}\right)^{\gamma^2-\gamma}\left(1 - \frac{2m}{\gamma r} + \frac{m^2\sin^2\theta}{\gamma^2 r^2}\right)^{1-\gamma^2} r^2\ ,\\
\tilde{g}_{\phi\phi} &= \left(1 - \frac{2m}{\gamma r}\right)^{1-\gamma} r^2 \sin^2\theta\ ,
\end{aligned} \tag{34}$$

where m is the mass of the object and γ is a deformation parameter of the spacetime. In the case when, $\gamma = 1$, the Schwarzschild spacetime can be reproduced. The Curzon–Chazy spacetime can be obtained in the following limit $\gamma \to \infty$. For $\gamma > 1$, the Zipoy–Voorhees geometry describes a spacetime surrounding a central object that is more oblate compared to a Schwarzschild black hole. Conversely, when $0 < \gamma < 1$, the central object takes on a more prolate shape. In the special case of $\gamma = 0$ (which is equivalent to setting $m = 0$), the spacetime reduces to Minkowski space. According to the no-hair theorem, when $0 < \gamma \neq 1$, the event horizon disappears, leading to the formation of a true curvature singularity at $r = 2m/\gamma$ in addition to the usual singularity at $r = 0$. As a result, the Zipoy–Voorhees metric does not describe a black hole but rather a naked singularity. This feature becomes especially evident when rewriting the solution in Weyl coordinates (ρ, z), which are naturally adapted to prolate spheroidal coordinates (x, y). These coordinates are commonly used to describe various singular and pathological solutions in general relativity, further emphasizing the distinct nature of the Zipoy–Voorhees metric. It is important to emphasize that the Kretschmann scalar reveals the presence of two singularities: one at $r = 0$ and the other at $r = 2m/\gamma$. Notably, in the limit $\gamma \to \infty$, particularly in the Curzon–Chazy spacetime, these singularities coincide, effectively merging into a single singularity at $r = 0$.

By applying the method outlined in Section 3, we obtain the expression

$$\Lambda = 1 + s^2\left[1 - \left(1 - \frac{2m}{\gamma r}\right)^{\gamma}\right]\ , \tag{35}$$

where the parameters m and s are determined from Eq. (29) as follows:

$$m = \frac{2M^2 - Q^2}{2M} = M - b\ , \tag{36}$$

$$s^2 = \frac{Q^2}{2M^2 - Q^2} = \frac{b}{M-b} \;, \tag{37}$$

where M being the total mass and $b = Q^2/2M$ is the charge parameter of the central object. Before deriving the final result, the radial coordinate should be transformed as $r \to r - 2b$, as was done in the previous section. Finally, the components of the charged Zipoy–Voorhees metric in the low-energy effective field theory for heterotic string theory can be found as

$$\begin{aligned}
g_{tt} &= -\left\{\frac{M}{M-b} - \frac{b}{M-b}\left[1-\frac{2(M-b)}{\gamma(r-2b)}\right]^{\gamma}\right\}^{-1}\left[1-\frac{2(M-b)}{\gamma(r-2b)}\right]^{\gamma} \;,\\
g_{rr} &= \left\{\frac{M}{M-b} - \frac{b}{M-b}\left[1-\frac{2(M-b)}{\gamma(r-2b)}\right]^{\gamma}\right\}\left[1-\frac{2(M-b)}{\gamma(r-2b)}\right]^{\gamma^2-\gamma-1}\left[1-\frac{2(M-b)}{\gamma(r-2b)}+\frac{(M-b)^2\sin^2\theta}{\gamma^2(r-2b)^2}\right]^{1-\gamma^2} \;,\\
g_{\theta\theta} &= \left\{\frac{M}{M-b} - \frac{b}{M-b}\left[1-\frac{2(M-b)}{\gamma(r-2b)}\right]^{\gamma}\right\}\left[1-\frac{2(M-b)}{\gamma(r-2b)}\right]^{\gamma^2-\gamma}\left[1-\frac{2(M-b)}{\gamma(r-2b)}+\frac{(M-b)^2\sin^2\theta}{\gamma^2(r-2b)^2}\right]^{1-\gamma^2}(r-2b)^2 \;,\\
g_{\phi\phi} &= \left\{\frac{M}{M-b} - \frac{b}{M-b}\left[1-\frac{2(M-b)}{\gamma(r-2b)}\right]^{\gamma}\right\}\left[1-\frac{2(M-b)}{\gamma(r-2b)}\right]^{1-\gamma}(r-2b)^2\sin^2\theta \;,
\end{aligned} \tag{38}$$

where γ is the deformation parameter of the spacetime along the mass M, and charge parameter b of the central object described by the charged Zipoy–Voorhees spacetime. Along the metric coefficients (38), the scalar and the vector fields in the low-energy effective field theory for heterotic string theory can be found as

$$\Phi = -\ln\left\{\frac{M}{M-b} - \frac{b}{M-b}\left[1-\frac{2(M-b)}{\gamma(r-2b)}\right]^{\gamma}\right\} \;, \tag{39}$$

$$A_t = \frac{\sqrt{2}Q}{M-b\left[1-\frac{2(M-b)}{\gamma(r-2b)}\right]^{\gamma}}\left\{1-\left[1-\frac{2(M-b)}{\gamma(r-2b)}\right]^{\gamma}\right\} \;. \tag{40}$$

Using surface integral, the total charge of the central object can be determined as

$$Q_{\text{tot.}} = \frac{1}{4\pi}\int_{S^2} e^{-\Phi} * F = 2\sqrt{2}Q \;. \tag{41}$$

In Ref. [1], the author noted that the charge undergoes a rescaling from $2\sqrt{2}Q$ to Q. Notice that dilaton charge $\mathcal{D}$ is identified as analogue Coulomb charge expansion of the dilaton field which can be found from the following expression:

$$\Phi(r) = \Phi_0 + \frac{\mathcal{D}}{r} + \mathcal{O}(r^{-2}) \;.$$

By expanding Eq. (39), one can easily find that dilaton charge $\mathcal{D} = 2b = Q^2/M$.

It is evident that the metric coefficients in (38) exhibit three singularities. The first singularity is located at the origin, $r = 2b$, while the second and third singularities are given by

$$r = \frac{2M}{\gamma} + \frac{2b(\gamma-1)}{\gamma} \;, \tag{42}$$

$$r = \frac{2M}{\gamma - \gamma\left(\frac{M}{b}\right)^{1/\gamma}} + 2b\left[1 + \frac{1}{\gamma\left(\frac{M}{b}\right)^{1/\gamma} - \gamma}\right] \;. \tag{43}$$

Moreover, a detailed analysis confirms that the Kretschmann scalar diverges at these singular points. However, due to the complexity of its explicit expression, we do not present it here.

Notice that when $b = 0$, the spacetime metric coefficients in (38) reduce to the Zipoy–Voorhees spacetime given by metric coefficients in (34), which represents a generalization of the Schwarzschild solution with an additional deformation parameter. Similarly, for $\gamma = 1$, the metric corresponds to the GMGHS spacetime, as given in (21), which describes a charged black hole solution in the context of low-energy string theory. Finally, in the special case where both $\gamma = 1$ and $b = 0$, the metric further simplifies to the well-known Schwarzschild spacetime, recovering the standard spherically symmetric, vacuum solution of Einstein's equations.

Let us also examine the special limiting case where $b = M$ (or equivalently, $Q = \sqrt{2}M$). Under this condition, the spacetime metric coefficients, scalar and vector fields take the following form:

$$g_{tt} = -\left(1 - \frac{2M}{r}\right) \;, \tag{44}$$

$$g_{rr} = \left(1 - \frac{2M}{r}\right)^{-1} \;, \tag{45}$$

$$g_{\theta\theta} = r(r-2M) \;, \tag{46}$$

$$g_{\phi\phi} = r(r-2M)\sin^2\theta \;, \tag{47}$$

$$\Phi = -\ln\left(1 - \frac{2M}{r}\right) \;, \tag{48}$$

$$A_t = 0\ , \tag{49}$$

which corresponds to a naked singularity, as discussed earlier. The Kretschmann scalar for above spacetime yields

$$K = \frac{4M^2\left(12r^2 - 32Mr + 27M^2\right)}{r^6(r-2M)^2}\ , \tag{50}$$

which contains two singularities at $r = 0$ and $r = 2M$.

Now, we can examine how the charged Zipoy–Voorhees metric deviates from the Schwarzschild spacetime for small values of the parameter q, defined as $q = \gamma - 1$. To do so, we expand the metric coefficients of the Zipoy–Voorhees spacetime as a power series in q. Finally, we obtain

$$g_{tt} = -\left(1-\frac{2M}{r}\right) - \frac{M(r-2b)}{r(M-b)}\left\{\frac{2(M-b)}{r} + \left(1-\frac{2M}{r}\right)\ln\left[1-\frac{2(M-b)}{r-2b}\right]\right\}q\ , \tag{51}$$

$$g_{rr} = \left(1-\frac{2M}{r}\right)^{-1} - \frac{q}{(M-b)(r-2M)^2}\left((2M-r)\left(2r(b-M)\ln\left(\frac{(b-M)^2\sin^2\theta - 2r(b+M) + 4bM + r^2}{(r-2b)^2}\right) + \right.\right. \tag{52}$$

$$\left.\left. + (2b(M-r) + Mr)\ln\left(\frac{r-2M}{r-2b}\right)\right) + 2M(b-M)(2b-r)\right)$$

$$g_{\theta\theta} = r(r-2b)\left[1 + q\left\{\frac{2(b-M)\left(\ln\left(\frac{(b-M)^2\sin^2(\theta) - 2r(b+M) + 4bM + r^2}{(r-2b)^2}\right) + \frac{b}{r}\right) + \left(\frac{2b}{r}(M-r) + M\right)\ln\left(\frac{2(b-M)}{r-2b} + 1\right)}{(M-b)}\right\}\right] \tag{53}$$

$$g_{\phi\phi} = r(r-2b)\sin^2\theta\left\{1 - \left\{\frac{2b}{r} + \frac{M(r-2b)}{r(M-b)}\ln\left[1-\frac{2(M-b)}{r-2b}\right]\right\}q\right\}\ . \tag{54}$$

5. Particle motion

Since the motion of particles serves as a fundamental method for examining the nature of spacetime, it offers valuable information about the metric's geometric and causal structure. The paths followed by test particles — whether timelike for massive objects or null for light — encode details about the curvature and symmetries of the spacetime background. By studying these trajectories, one can investigate the gravitational field, assess the accuracy of theoretical models, and gain insight into the consequences of alternative or extended gravity theories. For simplicity we consider the circular motion of test particle in the equatorial plane i.e., $\theta = \pi/2$. Applying the normalization condition of the four-velocity of test particle yields the following equation of motion:

$$\dot{t} = -\frac{\mathcal{E}}{g_{tt}(r)}\ , \tag{55}$$

$$\dot{\phi} = \frac{\mathcal{L}}{g_{\phi\phi}(r)}\ , \tag{56}$$

$$\dot{r}^2 = \mathcal{E}^2 - V_{\rm eff}(r)\ , \tag{57}$$

where $V_{\rm rm}(r)$ is effective for test particle in charged Zipoy–Voorhees metric defined as

$$V_{\rm eff}(r) = -g_{tt}(r)\left(\delta + \frac{\mathcal{L}^2}{g_{\phi\phi}(r)}\right)\ . \tag{58}$$

Here, $\mathcal{E}$ and $\mathcal{L}$ are the specific energy and angular momentum of particle. The parameter $\delta = 0$ corresponds to massless particles, while $\delta = -1$ refers to massive particles. Applying conditions $\dot{r} = 0$ and $\ddot{r} = 0$, the expressions for the critical values of the specific energy and the specific angular momentum can be obtained as

$$\mathcal{E}^2 = \frac{g_{tt}(r)^2 g'_{\phi\phi}(r)}{g_{tt}(r)g'_{\phi\phi}(r) - g_{\phi\phi}(r)g'_{tt}(r)}\ , \tag{59}$$

$$\mathcal{L}^2 = \frac{g_{\phi\phi}(r)^2 g'_{tt}(r)}{g_{tt}(r)g'_{\phi\phi}(r) - g_{\phi\phi}(r)g'_{tt}(r)}\ . \tag{60}$$

Photonsphere and shadow: The photonsphere and shadow of a black hole are a remarkable and significant feature in general relativity and alternative theories of gravity. The photonsphere represents a spherical region surrounding the black hole where the gravitational field is strong enough to allow photons (particles of light) to move along circular trajectories. This region marks the part of spacetime where gravity curves the path of light into closed circular orbits. However, such orbits are inherently unstable—any slight disturbance will either cause the photon to spiral into the black hole or escape to infinity. If the photonsphere is known for the given spacetime shadow radius of the compact object can be easily can be determined. In the case of a static, uncharged (Schwarzschild) black hole, the radius of the photonsphere is given by $r_{\rm ph} = 3M$, while the shadow radius is $r_{\rm Sh} = 3\sqrt{3}M$. It is also essential to study the photonsphere and shadow of the object within the charged Zipoy–Voorhees spacetime to understand how the

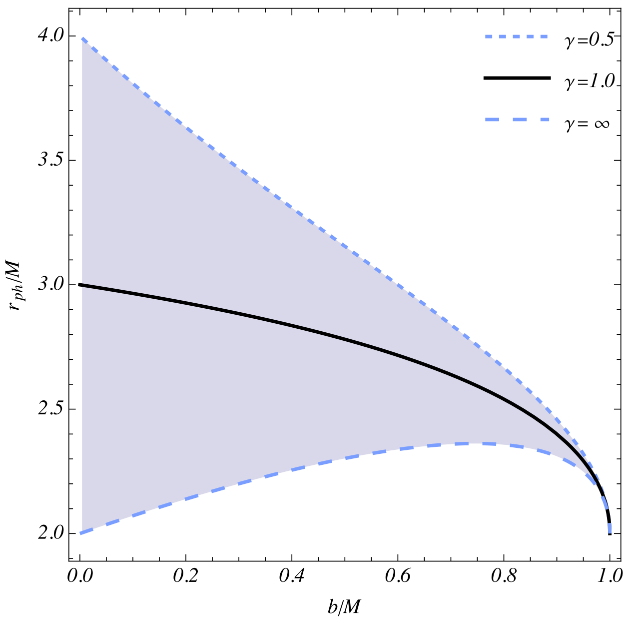

Fig. 1. Dependency of the photonsphere on the charge parameter b and deformation parameter γ. The solid line represents photonsphere in the Gibbons–Maeda–Garfinkle–Horowitz–Strominger spacetime.

electric charge and the deformation parameter influence the structure of the central object's shadow. To locate the photonsphere, one utilizes the condition that $\delta = 0$. So that radial equation reduces to the following form:

$$\frac{1}{\mathcal{L}^2}\dot{r}^2 = \frac{1}{r_{\rm Sh}^2} + \frac{g_{tt}(r)}{g_{\phi\phi}(r)} , \tag{61}$$

where $r_{\rm Sh} = \mathcal{L}/\mathcal{E}$ is the shadow radius of the compact object. Again one can use the following conditions $\dot{r} = \ddot{r} = 0$ to find the shadow and the photonsphere radii. Finally, one can obtain

$$r_{\rm Sh} = \sqrt{\frac{g_{\phi\phi}(r_{\rm ph})}{-g_{tt}(r_{\rm ph})}} , \tag{62}$$

where $r_{\rm ph}$ is the radius of the photonsphere found as a solution of equation $g_{tt}(r)g'_{\phi\phi}(r) - g_{\phi\phi}(r)g'_{tt}(r) = 0$. Mathematically, this corresponds to the following relation:

$$\left[1 - \frac{2(M-b)}{\gamma(r-2b)}\right]^{\gamma} = \frac{M[\gamma r + b - (1+2\gamma)M]}{b[\gamma r - M + b(1-2\gamma)]} . \tag{63}$$

Although obtaining an analytical solution to the above equation is difficult due to its complexity, numerical methods provide a viable approach to accurately determine the photonsphere radius. This enables a detailed investigation of the influence of various physical parameters on photon trajectories. A crucial aspect to emphasize is that the deformation parameter γ is constrained within the range $0.5 < \gamma < \infty$ (see, e.g., [23,51]). The lower bound, $\gamma > 0.5$, is imposed to ensure the existence of a shadow associated with the central compact object, which would otherwise vanish, undermining the physical interpretation of the spacetime geometry. On the other hand, the upper limit reflects the gradual transition from the charged Zipoy–Voorhees spacetime to the limiting case of the charged Curzon–Chazy solution, highlighting the continuous deformation in the spacetime structure governed by the γ parameter. In Fig. 1, we illustrate the dependence of the photonsphere radius on the charge parameter b for a range of values of the deformation parameter satisfying $1/2 < \gamma < \infty$. It is evident from the numerical results that, for fixed moderate values of γ, the presence of the charge parameter b leads to a noticeable decrease in the photonsphere radius. This behavior underscores the repulsive nature of the electromagnetic field, which alters the effective potential experienced by the photons and hence shifts the location of the unstable circular orbits. Interestingly, as the deformation parameter γ increases, the behavior of the photonsphere exhibits a non-monotonic trend. Specifically, for sufficiently large values of γ, the photonsphere radius initiates at $2M$, grows to reach a maximum at some intermediate value of the charge parameter, and subsequently decreases back to $2M$ as b continues to increase. This non-trivial behavior, shown in Fig. 1, reflects the interplay between gravitational, electromagnetic, and geometric effects encoded in the charged and deformed spacetime, and offers valuable insight into the structure of null geodesics in such configurations.

Next, we place constraints on the charge and deformation parameters by analyzing the shadow cast by the compact object described by the charged Zipoy–Voorhees spacetime. Based on Ref. [52], the observed shadow radius of M87* falls within the range:

$$4.31M \lesssim r_{\rm Sh} \lesssim 6.08M . \tag{64}$$

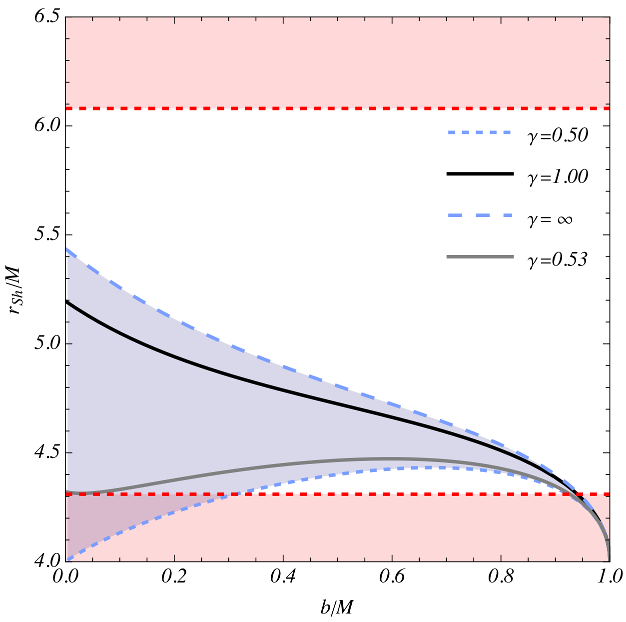

Fig. 2. Constraining parameters of the charged Zipoy–Voorhees spacetime from shadow of M87*. The light red region represent the excluded region of the shadow radius of M87*.

Our numerical analysis indicates that the shadow cast by the charged Zipoy–Voorhees object never reaches the upper bound of M87*'s shadow size. This suggests that the geometry of the spacetime, influenced by both the electric charge and the deformation parameter, plays a critical role in shaping the silhouette of the central object as observed by a distant observer. In Fig. 2, we illustrate how the shadow radius depends on the charge parameter b for several fixed values of the deformation parameter γ. As the figure shows, increasing the charge tends to reduce the size of the shadow. However, beyond a certain threshold—specifically, $b \gtrsim 0.91$—the predicted shadow radius falls below the lower limit of the observed range, effectively ruling out higher values of b. Thus, the shadow data imposes an upper bound on the charge parameter, constraining it to $b \lesssim 0.91$.

Moreover, by examining the interplay between the deformation parameter and the shadow size, we find that the deformation parameter γ must also satisfy a lower bound to remain consistent with observational data. From Fig. 2, one can observe that shadow sizes consistent with the lower observational bound ($r_{\rm Sh} \gtrsim 4.31M$) are only achievable when $\gamma \gtrsim 0.53$. This indicates that a minimal level of prolate deformation is required for the shadow to be observationally viable. Together, these constraints provide valuable insight into the physical characteristics of compact objects in alternative gravity models.

ISCO position: The Innermost Stable Circular Orbit (ISCO) is the smallest radius at which a test particle can stably orbit a massive object like a black hole. For familiar metrics like Schwarzschild or Kerr, ISCO is well-studied. However, for exotic or modified metrics such as the Zipoy–Voorhees spacetime, we need to define the metric explicitly before proceeding. In Refs. [23,29], it has been found that the ISCO position of test particle in the Zipoy–Voorhees spacetime is determined as

$$r_{\rm ISCO} = M\left(3+\frac{1}{\gamma}+\sqrt{5-\frac{1}{\gamma^2}}\right) . \tag{65}$$

It is well known that the ISCO position is located at stationary points of the critical value of the specific energy and specific angular momentum of particle. The radial derivative of Eqs. (59) and (60) leads to find the ISCO position. The solution of the following equation

$$\frac{g''_{tt}(r)}{g'_{tt}(r)}-\frac{2g'_{tt}(r)}{g_{tt}(r)}+\frac{2g'_{\phi\phi}(r)}{g_{\phi\phi}(r)}-\frac{g''_{\phi\phi}(r)}{g'_{\phi\phi}(r)}=0 , \tag{66}$$

leads finding the ISCO position of test particle in the charged Zipoy–Voorhees spacetime, where the metric coefficients g_{tt} and $g_{\phi\phi}$ are given in Eqs. (38). Finding the analytical solution of Eq. (66) is difficult; therefore we use numerical method to find dependence of the ISCO position from the charge and deformation parameters. In Fig. 3 we show dependence of the ISCO position of particle on the charge parameter b for $0.5 < \gamma < \infty$. It turns out that the ISCO position decreases as a result of the influence of the charge parameter.

Energy efficiency: The gravitational deflection of mass and energy efficiency of black holes are fascinating topics in astrophysics. A black hole's immense gravity bends spacetime, deflecting light and matter passing nearby. Matter often forms an accretion disk, spiraling inward while heating up and emitting radiation. Black holes convert matter into energy efficiently via accretion. As matter

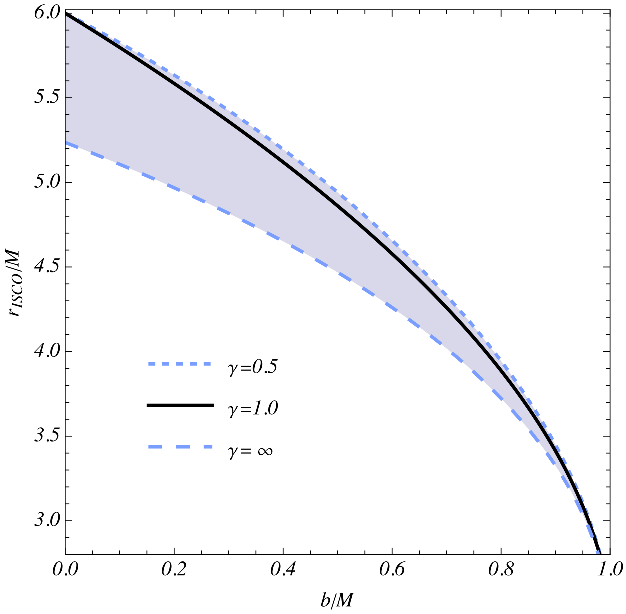

Fig. 3. The dependence of the ISCO position on the charge parameter b for $0.5 < \gamma < \infty$. The solid line represents the ISCO position of particle in the Gibbons–Maeda–Garfinkle–Horowitz–Strominger spacetime.

falls in, gravitational potential energy is released, mostly as heat and radiation in the accretion disk. According to Ref. [53], the energy efficiency of the black hole is determined by the following expression $\eta = 1 - \mathcal{E}_{ISCO}$. For the Schwarzschild black hole, the efficiency energy output relative to the mass energy input is about 5.7% while for an extreme Kerr black hole, it can reach up to 42% due to the ergosphere, where rotational energy can be extracted. Our numerical calculations show that the energy efficiency in the charged Zipoy–Voorhees spacetime reaches $\sim 14\%$. In Fig. 4 we demonstrate the dependence of the energy efficiency on the charge parameter for $0.5 < \gamma < \infty$.

6. Conclusions

In this study, we has obtained the charged Zipoy–Voorhees spacetime using the Hassan–Sen transformation, a robust and systematic technique for generating charged black hole solutions within the framework of low-energy heterotic string theory, starting from neutral vacuum solutions in general relativity. By exploiting the hidden symmetries and duality properties inherent to string theory, this transformation enables the construction of complex spacetimes enriched with additional physical fields, specifically the string-frame metric $G_{\mu\nu}$, the Kalb–Ramond field $B_{\mu\nu}$, the Maxwell field A_μ, and the dilaton field Φ. These fields collectively describe a charged spacetime configuration that extends beyond the classical general relativistic framework, incorporating string-theoretic corrections that influence the geometry and dynamics of the resulting black hole solutions. The Hassan–Sen transformation leverages the interplay between the neutral solution's structure and the string theory fields to introduce electric charge in a consistent manner, preserving the underlying symmetries of the theory. This approach not only provides a pathway to generate new solutions but also offers insights into the deeper connections between general relativity and string theory, particularly in how charge and additional fields modify the spacetime's properties.

We employed the Hassan–Sen transformation on well-known vacuum solutions, like the Schwarzschild and Zipoy–Voorhees metrics, showcasing its flexibility. By applying it to the Schwarzschild spacetime, we obtained the Sen or Gibbons–Maeda–Garfinkle–Horowitz–Strominger (GMGHS) metric, incorporating charge and dilaton effects while maintaining the event horizon at $r_h = 2M$. This result illustrates the transformation's power to bridge classical general relativity with string theory, altering thermodynamic properties—such as entropy, which becomes zero in the extremal case ($Q^2 = 2M^2$). Likewise, we transformed the Zipoy–Voorhees metric to derive its charged version, defined by a deformation parameter γ. This solution exhibits a complex singularity structure, including naked singularities when $\gamma \neq 1$, highlighting the transformation's ability to describe unconventional gravitational phenomena beyond standard black holes.

Our analysis of the photonsphere and shadow in the charged Zipoy–Voorhees spacetime reveals profound insights into the interplay of electric charge b, deformation parameter γ. Numerical results are demonstrated that increasing charge reduces the photonsphere radius, reflecting the repulsive electromagnetic influence, while large γ values introduce non-monotonic behavior, with the radius peaking at intermediate charges before returning to $2M$. This highlights the intricate balance of gravitational, electromagnetic, and geometric effects. Furthermore, by comparing the shadow radius to the observed M87* shadow ($4.31M \lesssim r_{\rm Sh} \lesssim 6.08M$), we establish stringent constraints: the charge parameter is bounded by $b \lesssim 0.91$, and the deformation parameter requires

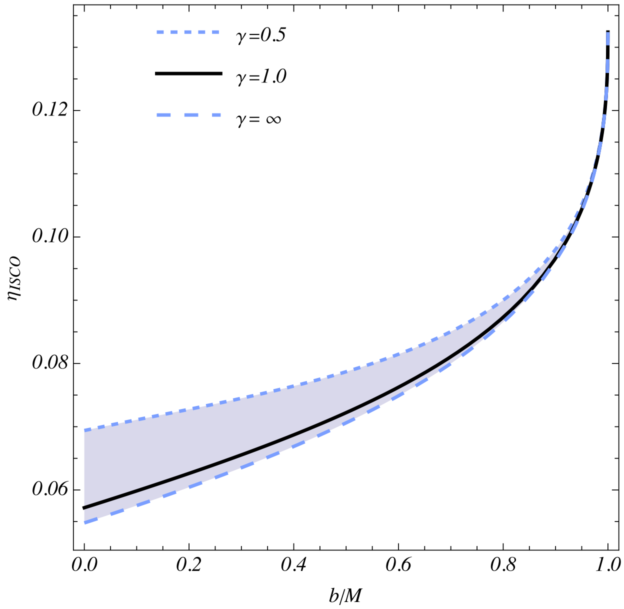

Fig. 4. Energy efficiency for charged gamma metric.

$\gamma \gtrsim 0.53$ to produce observationally viable shadow sizes. These bounds underscore the critical role of spacetime deformation and charge in shaping observable signatures of compact objects. While the charged Zipoy–Voorhees spacetime never attains the upper M87* shadow limit, its study enriches our understanding of null geodesics and shadow formation in alternative gravity frameworks, offering a valuable bridge between theoretical predictions and astrophysical observations.

We have studied massive particle motion in the charged Zipoy–Voorhees spacetime. Investigation of the ISCO and energy efficiency in the charged Zipoy–Voorhees spacetime is provided significant insights into the dynamics of test particles and energy processes around exotic compact objects. The ISCO position, determined numerically due to the complexity of the governing equations, reveals a clear dependence on the charge parameter b and deformation parameter γ. Specifically, the ISCO radius decreases with increasing charge, indicating the influence of the charge in tightening stable orbits. Furthermore, the analysis of the energy efficiency showed that the charged Zipoy–Voorhees spacetime achieves an efficiency of approximately 14%, surpassing the Schwarzschild black hole's 5.7% but falling short of the extreme Kerr black hole's 42%. The dependence of energy efficiency on the charge parameter highlights how charge and deformation enhance the conversion of gravitational potential energy into radiation within the accretion disk.

CRediT authorship contribution statement

Odil Yunusov: Visualization, Software, Resources, Methodology, Investigation, Conceptualization. **Bobur Turimov:** Writing – original draft, Investigation, Conceptualization. **Yokubjon Khamroev:** Methodology, Investigation, Formal analysis, Conceptualization. **Sulton Usanov:** Visualization, Validation, Investigation. **Farkhodjon Turaev:** Validation, Methodology, Formal analysis.

Declaration of competing interest

Authors have no conflicts of interest to disclose.

Data availability

No data was used for the research described in the article.

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